# Johns Hopkins Math Tournament 2021

## Proof Round B - Middle School & High School

# Protecting Polygons

*April 3, 2021*

## Instructions

- To receive full credit, answers must be legible, orderly, clear, and concise.
- Even if not proven, earlier numbered items may be used in solutions to later numbered items, but not vice versa.
- Middle school teams are not required to solve the problems designated "HIGH SCHOOL ONLY."
- While this round is asynchronous, you are still NOT ALLOWED to use any outside resources, including the internet, textbooks, or other people outside your teammates.
- Put the **team number** (NOT the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.
- To submit your answers, please email ONE SINGLE PDF containing all your answers to jhmt2021proofroundB@gmail.com, with the subject tag as "Team # Proof Round B". For example, if your team number is 0, then the subject should be "Team 0 Proof Round B".

# Introduction

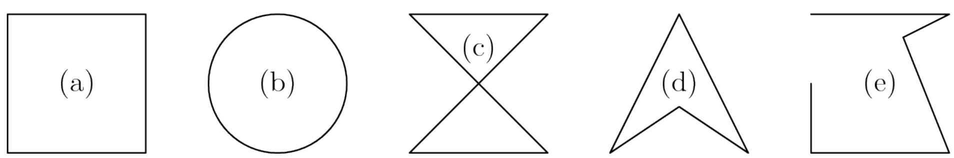
Rich Richard wants to store his fortune at "Fort Knoxagon," a polygonal shaped building. He is concerned about security, so he wants to hire some guards. As an initial plan, he decided to assign guards to each of the vertices of Fort Knoxagon (we will assume that the guards can see in every direction). However, this plan was not sustainable; in order to save money, he wants to find the minimum number of guards needed so that Fort Knoxagon is still secure. Throughout this proof round, we will help Rich Richard solve this problem by investigating topics relating to geometry and combinatorics.

# 1 Polygons and Convexity

**Definition 1.1.** A *polygon* is a (closed) region in $\mathbb{R}^2$ (i.e. the coordinate plane) that is bounded by finitely many line segments; it forms a (closed) curve that is not self-intersecting. The *edges* of the polygon are the line segments, and the *vertices* of the polygon are the intersection points between adjacent edges.

**Remark 1.1.** All points on the perimeter of a polygon or contained within the interior of the polygon are inside the polygon.

**Problem 1:** (10 points) Identify which of the following are polygons. For each figure that is not a polygon, explain briefly why it is not a polygon.




Only (a) and (d) are polygons. (b), (c), and (e) are not polygons; (b) is not bounded by finitely many line segments, (c) is self-intersecting, and (e) is not closed.

**Definition 1.2.** A polygon $\mathcal{P}$ is *convex* if for each pair of points $(\mathbf{p_1}, \mathbf{p_2}) \in \mathcal{P}$ and any $\lambda \in [0,1]$, $\lambda\mathbf{p_1} + (1-\lambda)\mathbf{p_2} \in \mathcal{P}$. The notation $\mathbf{p} \in \mathcal{P}$ means that the point $\mathbf{p}$ is inside or on the perimeter of $\mathcal{P}$. Similarly, the notation $\lambda \in [0,1]$ means that $\lambda$ is a real number between 0 and 1 inclusive.

**Remark 1.2.** The above definition is equivalent to stating that if we draw a line segment between any two points in $\mathcal{P}$, then that line segment will be contained (entirely) inside or on the perimeter of $\mathcal{P}$.

**Problem 2:** (5 points) Of the polygons identified in Problem 1, which are convex? If the polygon is not convex, briefly explain why.

(a) is convex and (d) is not convex. (d) is not convex because a line segment between its bottom two vertices lies outside of the polygon.

We did not accept answers that specified whether (b), (c), and/or (e) were convex, because the problem asked for which of the figures were convex polygons, and the aforementioned figures are not polygons, as stated in the solution to Problem 1.

**Problem 3:** (15 points) Using Definition 1.2, prove that the square with vertices

$$(1,-1),\ (1,1),\ (-1,1),\ \text{and}\ (-1,-1)$$

is convex.

Let us name the square specified in the problem $\mathcal{S}$. First, notice that in general, a point $(x_0,\ y_0) \in \mathcal{S}$ if $-1 \leq x_0 \leq 1$, and $-1 \leq y_0 \leq 1$.

Now, consider arbitrary points $\mathbf{p_1} = (x_1,\ y_1)$ and $\mathbf{p_2} = (x_2,\ y_2)$ such that $\mathbf{p_1}, \mathbf{p_2} \in \mathcal{S}$. For some $\lambda \in [0, 1]$, let

$$\mathbf{p_3} = \lambda \mathbf{p_1} + (1-\lambda)\mathbf{p_2} = (\lambda x_1 + (1-\lambda)x_2,\ \lambda y_1 + (1-\lambda)y_2) = (x_3,\ y_3)$$

Now, since $x_3 = \lambda x_1 + (1-\lambda)x_2$, let us consider the possible range of values $x_3$ can take. Since $\lambda$ and $1-\lambda$ are both non-negative numbers, $x_3$ is maximized when $x_1$ and $x_2$ are both as large as possible, i.e. $x_1 = x_2 = 1$. In this case, $x_3 = \lambda + (1 - \lambda = 1$, i.e. the largest possible value of $x_3$ is 1. Next, $x_3$ is minimized when $x_1$ and $x_2$ are both as small as possible, i.e. $x_1 = x_2 = -1$. In this case, $x_3 = -\lambda - (1-\lambda) = -1$, i.e. the smallest possible value of $x_3$ is $-1$. Thus, we find that $-1 \leq x_3 \leq 1$. This exact argument can also be applied to $y_3$ to show that $-1 \leq y_3 \leq 1$, thus by our initial observation on points $\in \mathcal{S}$, $(x_3,\ y_3) = \mathbf{p_3} \in \mathcal{S}$. Since the entire argument above used arbitrary values for $\mathbf{p_1}$, $\mathbf{p_2}$, and $\lambda$, we can conclude that $\mathcal{S}$ is convex.

Another possible solution uses the fact that for any real numbers $a$, $b$, $|a+b| \leq |a| + |b|$, and $|ab| = |a||b|$. These two facts can be used to show:

$$|x_3| = |\lambda x_1 + (1-\lambda)x_2| \leq |\lambda x_1| + |(1-\lambda)x_2| = \lambda|x_1| + (1-\lambda)|x_2| \leq \lambda + (1-\lambda) = 1.$$

In other words, $|x_3| \leq 1$, and since the same argument can be applied to $y_3$ to show $|y_3| \leq 1$, we have successfully shown that $\mathbf{p_3} \in \mathcal{S}$.

**Problem 4:** (5 points) Show that if a polygon has an angle with measure greater than $180°$, then it is not convex.

Suppose that in polygon $\mathcal{P}$, $\angle ABC$ is an angle of measure greater than $180°$. Thus, we claim that the line segment formed by connecting vertex $A$ and vertex $C$ is outside of $\mathcal{P}$. If this claim is true, then it immediately follows that $\mathcal{P}$ is not convex by the definition of convexity.

We will prove this claim by contradiction, thus for sake of contradiction, assume that $\overline{AC} \in \mathcal{P}$. Since $A$, $B$, and $C$ are consecutive vertices, $\triangle ABC$ must be a triangle completely contained inside $\mathcal{P}$, since $\overline{AB}$ and $\overline{BC}$ are sides of $\mathcal{P}$ (i.e. $\overline{AC}, \overline{BC}, \overline{AC} \in \mathcal{P}$ was assumed). Thus, as with any triangle, the angle measures of $\triangle ABC$ must sum to $180°$. However, since $\angle ABC > 180°$, the sum of the angles of $\triangle ABC$ must sum to more than $180°$, which is absurd. Therefore, it cannot be the case that $\overline{AC} \in \mathcal{P}$, and thus we have proven our claim.

## 2 Triangulations

**Definition 2.1.** A *diagonal* of a polygon $\mathcal{P}$ is a line segment $\ell$ that satisfies the following properties:

- $\ell$ is contained in the interior of $\mathcal{P}$.
- $\ell$ is not an edge of $\mathcal{P}$.
- the endpoints of $\ell$ are two vertices of $\mathcal{P}$.

**Definition 2.2.** Two diagonals of a polygon are *noncrossing* if they do not intersect at an interior point of the polygon.

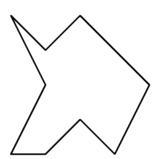
**Remark 2.1.** If two distinct diagonals share an endpoint, they are noncrossing.

**Theorem 2.1.** *Any polygon $\mathcal{P}$ with more than 3 vertices has a diagonal.*

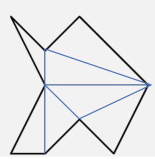
*Proof.* Toss $\mathcal{P}$ in the coordinate plane. Pick the vertex $V$ with the smallest $y$-coordinate (if there is more than one, pick the vertex with the largest $x$-coordinate), and consider vertices $A$ and $B$ such that $\overline{VA}$ and $\overline{VB}$ are edges of $\mathcal{P}$. Draw $\overline{AB}$. Either it is a diagonal of $\mathcal{P}$ and we are done, or it is not. If $\overline{AB}$ is not a diagonal, because $\mathcal{P}$ has more than 3 vertices, $\triangle ABV$ must contain some vertex of $\mathcal{P}$. Now, draw a line $\ell$ parallel to $\overline{AB}$ passing through $V$, and slide it up parallel to itself until it hits a vertex of $\mathcal{P}$. Let $X$ be this vertex. Then, the region of $\mathcal{P}$ below $\ell$ and above $V$ contains no vertices of $\mathcal{P}$, so $\overline{VX}$ is a diagonal of $\mathcal{P}$ by definition. $\square$

**Definition 2.3.** Let $\mathcal{P}$ be a polygon. A *triangulation* of $\mathcal{P}$ is a dissection of $\mathcal{P}$ into triangles by a maximal set of noncrossing diagonals. We define "maximal" to mean that no more diagonals of $\mathcal{P}$ can be drawn without crossing another diagonal in the set.

**Problem 5:** (5 points) Draw a triangulation of the following polygon:



Any solution that draws non-intersecting diagonals of the polygon to split it into 7 triangles is a valid solution. Here is one example below:



**Problem 6:** (10 points) Prove that every polygon has a triangulation. Hint: use induction. HIGH SCHOOL ONLY.

We will induct over the number of vertices $n$ in the polygon $\mathcal{P}$. Thus, the base case will be for $n = 3$. Since triangles do not have diagonals, an unchanged triangle is a valid triangulation since it is already composed of exactly 1 triangle. Next, using the strong inductive hypothesis, assume that all polygons with $n \leq k$ vertices have a triangulation.

Now, consider a polygon $\mathcal{P}$ with $n = k+1$ vertices. By Theorem 2.1 above, $\mathcal{P}$ must have a diagonal. Thus, we draw that diagonal of $\mathcal{P}$ and split $\mathcal{P}$ into two distinct but adjacent polygons $\mathcal{Q}$ and $\mathcal{R}$. Notice that $\mathcal{Q}$ must have at least 1 vertex that is not shared with $\mathcal{R}$, because otherwise $\mathcal{Q}$ would not be distinct from $\mathcal{R}$. Similarly, $\mathcal{R}$ must have at least 1 vertex that is not shared with $\mathcal{Q}$. Thus, both $\mathcal{Q}$ and $\mathcal{R}$ have less than $k+1$ vertices, and by the strong inductive hypothesis, $\mathcal{Q}$ and $\mathcal{R}$ must have triangulations. Since $\mathcal{Q}$ and $\mathcal{R}$ are adjacent but non-intersecting, their combined triangulations form a triangulation for $\mathcal{P}$, and we have finished our proof by induction.

Lastly, note that it is NOT sufficient to say that drawing an arbitrary diagonal of $\mathcal{P}$ will necessarily split $\mathcal{P}$ into a triangle and another polygon, because if $\mathcal{P}$ is not convex, some diagonals are not possible to draw as they will not be inside the $\mathcal{P}$. Thus, valid solutions must either prove that there exists at least one diagonal splits $\mathcal{P}$ into a triangle, or do what the solution above does and not assume that a diagonal splits $\mathcal{P}$ into any specific polygon.

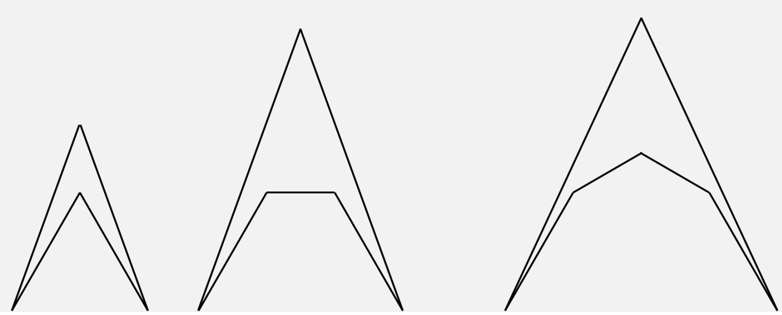
**Problem 7:** (10 points) For each $n > 3$, construct a polygon with $n$ vertices that has a unique triangulation; justify why the construction works.

The figures below show examples of such polygons for $n = 4$, $n = 5$, and $n = 6$. The same strategy shown in these examples can be extended for any $n$. In particular, to construct an $n$-gon with a unique triangulation, one can draw the upper half of a regular $(2n - 4)$-gon, choose a point far enough above the half, and then draw line segments connecting the lowest 2 vertices of the half to that point.

Note that this strategy for polygon construction results in unique triangulations because in all the polygons, the only valid diagonals are line segments that connect the vertex at the top of the polygon to the vertices not adjacent to the top vertex (i.e. the vertices below the top vertex, except for the two vertices adjacent to the top vertex); there are $n - 3$ such line segments. No other diagonals are possible because any other method of connecting non-adjacent vertices results in line segments that are outside the polygon. To see this in more rigor, given such an $n$-gon, there are $\binom{n}{2}$ line segments that can be drawn between 2 of its vertices. But 2 of these line segments are the outer edges of the polygon. Moreover, $\binom{n-1}{2}$ of these line segments connect vertices of the upper half of a regular $(2n - 4)$-gon, which are either edges of the polygon or line segments outside of the polygon. Thus,

$$\binom{n}{2} - \binom{n-1}{2} - 2 = \frac{n(n-1) - (n-1)(n-2)}{2} - 2 = n - 3$$

of the line segments are actually diagonals of the polygon, as we claimed earlier. By the nature of our construction, these diagonals are noncrossing, and we must draw all of them to obtain a valid triangulation; therefore, the polygon has a unique triangulation.



# 3 But What About Fort Knoxagon?

**Definition 3.1.** Consider a point $\mathbf{p}$ inside Fort Knoxagon. A guard can *see* $\mathbf{p}$ if the line segment connecting the point at which the guard is standing and $\mathbf{p}$ lies entirely inside the polygon defined by the fort.

**Definition 3.2.** A guard's *field of view* is the set of all points that they can see.

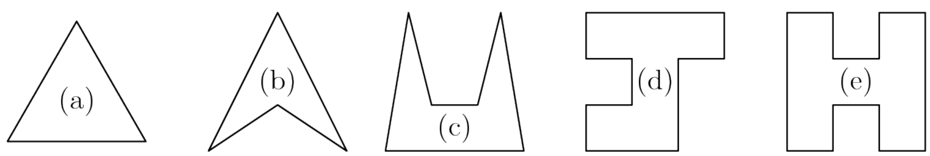
**Remark 3.1.** A guard's field of view is restricted by the edges of the polygon.

**Definition 3.3.** Suppose that there are $n$ guards defending Fort Knoxagon. Let these guards have field of views $\mathcal{V}_1, \dots, \mathcal{V}_n$, and let $\mathcal{F}$ denote the polygon defined by Fort Knoxagon. We say that Fort Knoxagon is *protected* if

$$\bigcup_{i=1}^{n} \mathcal{V}_i = \mathcal{F}.$$

That is, Fort Knoxagon is protected if the collective fields of view of the guards is $\mathcal{F}$.

**Problem 8:** (10 points) For the polygons below, find the minimum number of guards needed to protect each one; provide an explanation for (c).




(a) 1

(b) 1

(c) 1 or 2 depending on explanation

(d) 2

(e) 2

**Problem 9:** (10 points) Prove that every convex polygon can be protected by 1 guard.

Let $\mathcal{P}$ be a convex polygon, and suppose we situate our 1 guard at point $\mathbf{p}$ such that $\mathbf{p} \in \mathcal{P}$. Now, for the sake of contradiction, assume that $\mathcal{P}$ is not protected by this one guard. This implies that there exists at least 1 point $\mathbf{q}$ such that the guard at $\mathbf{p}$ cannot see $\mathbf{q}$, i.e. $\overline{\mathbf{pq}} \notin \mathcal{P}$. However, since, $\mathbf{p}, \mathbf{q} \in \mathcal{P}$, this means that by the definition of polygon convexity, $\mathcal{P}$ is not a convex polygon. As a result of this contradiction, our initial assumption must be false, which means that $\mathcal{P}$ *can* be protected by 1 guard.

And finally, the following problem will use all of the concepts that we have learned so far!

**Problem 10:** (20 points) Let $\mathcal{F}$ have $n$ vertices. As a function of $n$, find and prove an upper bound on the number of guards needed to protect $\mathcal{F}$ (this bound should be as strong as possible and sufficient for any $\mathcal{F}$).
HIGH SCHOOL ONLY.

The desired bound is $\left\lfloor \frac{n}{3} \right\rfloor$. To prove that this is sufficient for any $\mathcal{F}$, consider a triangulation of $\mathcal{F}$. It is possible to color each of the vertices of $\mathcal{F}$ with 1 of 3 colors such that each "triangle" in the triangulation has distinctly-colored vertices (i.e. there exists a 3-coloring of the triangulation); this can be shown with either induction or graph theory. By the Pigeonhole Principle, the least common color in the coloring appears $\left\lfloor \frac{n}{3} \right\rfloor$ times; situate a guard there at each vertex that is assigned that color. Then, since each triangle of the triangulation is protected by a guard (due to the convexity of triangles), the collective fields of view of the guards is $\mathcal{F}$, so $\mathcal{F}$ as a whole is protected.

To show that this bound is as strong as possible, we will demonstrate that $\left\lfloor \frac{n}{3} \right\rfloor$ guards are needed to protect some polygons. Consider, for instance, the hexagon depicted in Problem 8c with 2 "spikes." If the line segment connecting the spikes is sufficiently long, then 2 guards are needed to protect the polygon (we situate the guards along the bottom to protect each spike). We may construct similar polygons with $n$ spikes; each polygon will have $3n$ edges (2 edges for each spike, $n-1$ edges to connect the spikes, and 1 edge at the bottom). Thus, $\left\lfloor \frac{n}{3} \right\rfloor$ guards are needed to protect some $n$-gons, as claimed.

Hopefully, this will save Rich Richard enough money!

**Author's Note:** The main problem I introduced during this round is a depiction of the Art Gallery Problem. I hope that you enjoyed deriving Professor Steve Fisk's brilliant solution to this problem!

If you want to dive deeper into the concepts covered during this round, I encourage you to check out *Discrete and Computational Geometry* by Devadoss and O'Rourke, which I used as a reference while writing this round; it's a very neat book!

Lastly, I want to thank RK for writing many of the solutions above, as well as Ex Numera for proofreading the round.